arXiv:gr-qc/0102033v1 8 Feb 2001

On the linear stability of solitons and hairy black holes with a negative cosmological constant: the odd-parity sector

O. Sarbach⋆ and E. Winstanley†
⋆Center for Gravitational Physics and Geometry,
Department of Physics, The Pennsylvania State University,
104 Davey Laboratory, University Park, PA 16802.
†Department of Applied Mathematics, The University of Sheffield,
Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

February 7, 2008

Abstract

Using a recently developed perturbation formalism based on curvature quantities, we investigate the linear stability of black holes and solitons with Yang-Mills hair and a negative cosmological constant. We show that those solutions which have no linear instabilities under odd- and even-parity spherically symmetric perturbations remain stable under odd-parity, linear, non-spherically symmetric perturbations.

1 Introduction

The discovery of solitonic [1] and black hole [2] solutions to $\mathfrak{su}(2)$ Einstein-Yang-Mills (EYM) theory sparked considerable study of the properties of non-Abelian gauge theories coupled to gravity (see [3] for a comprehensive review of the subject and a full list of references). Many examples of both globally regular and black hole solutions have now been discovered, in a wide range of theories involving various non-Abelian gauge fields, both with and without a (positive or negative) cosmological constant.

Of this plethora of examples, many (but not all) are unstable. In particular, there is a general result that solitons and black holes solutions of EYM theory with a compact gauge group in asymptotically flat space must be topologically unstable [4]. This instability is analogous to the instability of the flat-space Yang-Mills-Higgs sphaleron [5]. In addition, $\mathfrak{su}(2)$ EYM solitons and black holes in the presence of a positive cosmological constant are unstable [6]. It was therefore surprising to discover that this result does not extend to solutions in anti-de Sitter (adS) space, when there is a negative cosmological constant.

Both soliton [7] and black hole [8] solutions have been found which are linearly stable with respect to spherically symmetric perturbations. It is the purpose of the present article to examine the linear stability of these solutions with respect to non-spherical perturbations.

We concentrate here on the odd-parity (or sphaleronic) sector. For spherically symmetric perturbations, the behaviour in this sector is "topological" in the sense that it does not depend on the detailed structure of the solutions, but only on global properties and boundary conditions. Further, it is in this sector (which, for spherically symmetric perturbations, involves the perturbations of the gauge field only, and not the metric perturbations) in which the analogy with the flat space sphaleron is most pertinent. Analysis of this sector for solitons and black holes in asymptotically flat space has shown that all the modes of instability are contained within the spherically symmetric perturbations [9]. Again, this is the same as for the flat space sphalerons, where the only modes of instability are spherically symmetric [10]. In this article we extend this result to solitons and black holes in asymptotically anti-de Sitter space, and show analytically that those solutions which are stable under spherically symmetric perturbations remain stable under non-spherically symmetric perturbations in the odd-parity sector. We shall return to the question of the stability under non-spherically symmetric perturbations in the even-parity sector in a future publication.

In order to discuss the stability, we use a recently developed perturbation formalism, which is based on curvature quantities. The main advantage of this formalism is that it allows us to cast the pulsation equations, governing linear fluctuations on a static background, into the form of a gauge invariant wave equation even when complicated matter fields are coupled to the metric. It has been shown in [11] that for a static and purely magnetic solution of the EYM equations (with arbitrary compact gauge group), the pulsation equations admit the form of a symmetric wave equation for the linearized extrinsic curvature and the electric field. It is precisely this symmetric form of the pulsation equations which will be crucial in this article to show the stability of the above mentioned solitons and black holes by analytic means.

This work is organized as follows. In section 2 we remember some important results about the hairy black hole solutions with a negative cosmological constant, recently found in [8], and the corresponding solitonic solutions [7]. In section 3, we briefly review the curvature-based formalism of perturbation theory for a static background, and also generalize to the case where a cosmological constant is present. The harmonic decomposition and the decoupling of the constraint and dynamical variables are performed in section 4, where the absence of exponentially growing modes is also shown. Technical details and the initial value formulation in terms of gauge-invariant quantities are discussed in Appendix A and B, respectively.

The metric signature is $(-,+,+,+)$ throughout, and we use the standard notations $2\omega_{(ab)} = \omega_{ab} + \omega_{ba}$ and $2\omega_{[ab]} = \omega_{ab} - \omega_{ba}$ for symmetrizing and

antisymmetrizing, respectively. Throughout the paper, greek letters denote spacetime indices taking values in $(0,1,2,3)$, while roman letters will denote spatial indices taking values $(1,2,3)$.

2 Solitons and hairy black holes with a negative cosmological constant

Hairy black holes in $\mathfrak{su}(2)$ Einstein-Yang-Mills theory with a negative cosmological constant were first found in [8], and subsequently in [7], where the corresponding solitonic solutions are also discussed. The equilibrium metric is spherically symmetric

$$ds^2 = -N(r)S^2(r)\,dt^2 + N^{-1}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),$$

and the gauge field potential has the spherically symmetric form

$$A = (1 - w(r))\left[-\tau_\phi d\theta + \tau_\theta \sin\theta\,d\phi\right].$$

Here the $\mathfrak{su}(2)$ generators $\tau_{r,\theta,\phi}$ are given in terms of the usual Pauli matrices σ_i by $\tau_r = \underline{e}_r \cdot \underline{\sigma}/2i$, etc. Writing $N(r) = 1 - 2m(r)/r - \Lambda r^2/3$, where Λ is the (negative) cosmological constant, the field equations take the form:

$$\begin{aligned} m_r &= G\left[N w_r^2 + \frac{1}{2r^2}(w^2-1)^2\right], \\ \frac{S_r}{S} &= \frac{2Gw_r^2}{r}, \\ 0 &= Nr^2 w_{rr} + \left(2m - \frac{2\Lambda r^3}{3} - G\frac{(w^2-1)^2}{r}\right) w_r + (1-w^2)w, \end{aligned} \tag{1}$$

where G is Newton's constant, and we have set the gauge coupling constant equal to $\sqrt{4\pi}$ for convenience.

We are interested in solutions which approach anti-de Sitter (adS) space at infinity, so that the asymptotic behaviour of the field functions is:

$$\begin{aligned} m(r) &= M + \frac{M_1}{r} + O(r^{-2}), \\ w(r) &= w_\infty + \frac{w_1}{r} + O(r^{-2}), \\ S(r) &= 1 + O(r^{-4}). \end{aligned}$$

Here we already observe one difference between the configurations in asymptotically flat, and asymptotically adS space. For asymptotically flat space solutions, either $w_\infty = 0$, in which case the Reissner-Nordström (RN) solution follows, or $w_\infty = \pm 1$, so that there is no magnetic charge at infinity [12]. However, in adS, the boundary conditions place no restriction on the value of w_∞,

so in general, even non-Abelian solutions will be globally magnetically charged. For black hole solutions having a regular event horizon at $r = r_h$, all the field variables have regular Taylor expansions near the event horizon, for example

$$w(r) = w(r_h) + w_r(r_h)(r - r_h) + O(r - r_h)^2.$$

However, there are just two independent parameters, $S(r_h)$ and $w(r_h)$ since $N = 0$ at the event horizon, which gives

$$m(r_h) = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6}.$$

In order for the event horizon to be regular, we shall also require that $N_r(r_h) > 0$, which implies that

$$F_h \equiv 1 - \Lambda r_h^2 - G\frac{(w(r_h) - 1)^2}{r_h^2} > 0.$$

From (1), one has

$$w_r(r_h) = \frac{(w(r_h)^2 - 1)w(r_h)}{r_h F_h}.$$

There are also globally regular (solitonic) solutions, for which the behaviour near the origin is:

$$\begin{aligned} m(r) &= 2Gb^2r^3 + O(r^4), \\ w(r) &= 1 - br^2 + O(r^3), \\ S(r) &= S(0)\left[1 + 4Gb^2r^2 + O(r^3)\right]. \end{aligned}$$

Here the independent parameters are b and $S(0)$.

The simplest solutions to the field equations (1) are the Schwarzschild-adS solution,

$$w = 1, \quad S = 1, \quad m = \text{const}.$$

and the RN-adS solution

$$w = 0, \quad S = 1, \quad m = \text{const.} - \frac{G}{2r}.$$

In both cases, the YM field is effectively Abelian. It is however interesting to study the stability of these solutions with respect to non-Abelian perturbations.

The solutions of greatest interest in this article are effectively non-Abelian solutions, for which the gauge function w has no zeros, since these solutions were shown in [8] to be linearly stable to both even and odd-parity spherical perturbations. These solutions are unique to anti-de Sitter space, as w must have at least one zero if the cosmological constant is positive or zero [13, 14]. In [8] it is proved that for any value of the gauge field at the event horizon, $w(r_h) \neq 0$, for all sufficiently large $|\Lambda|$ there is a black hole solution in which w

has no zeros. Similar behaviour is found numerically for the solitonic solutions [7]. For spherical perturbations of these equilibrium configurations, it is proved analytically that all the solutions in which w has no zeros are stable in the odd-parity sector [8]. The even-parity sector is more complicated, but stability can be proven for sufficiently large $|\Lambda|$.

In section 4 we shall require various properties of these soliton and black hole solutions, particularly when $|\Lambda| \gg 1$. We now briefly review the relevant results, and refer the reader to [7, 8] for further details and proofs. We are concerned with black hole and soliton solutions in which the gauge field function w has no zeros, which exist for sufficiently large $|\Lambda|$. As $|\Lambda| \to \infty$, the function w approaches a constant value, given by $w(r_h)$ for black hole solutions, and $w \equiv 1$ for solitons. This means that $w_r(r)$ tends to zero for all r as $|\Lambda| \to \infty$. In [8], it was shown that in fact

$$w_r(r) \sim o(|\Lambda|^{-\frac{1}{2}}) \qquad \text{as } |\Lambda| \to \infty$$

for the black hole solutions, and that proof is easily extended to show that the same is also true for the solitons. This result will be useful in section 4.

The other result we shall require in section 4 is that for these solutions in which w has no zeros, the gauge function w cannot be equal to ± 1 anywhere, with the exception of the origin for the solitons (where regularity conditions insist that $w = \pm 1$) and the Schwarzschild-adS solution. To see this, suppose that $w > 0$ everywhere. Then w cannot have a local minimum in the region $0 < w < 1$ nor a local maximum in the region $w > 1$ [8, 13]. For regular solutions, $w = 1$ at the origin, and if it is initially increasing, it is necessary for w to first reach a local maximum if it is to cross $w = 1$ again, which cannot happen. Similarly, if w is initially decreasing, then, because w has no zeros, it cannot have a local minimum which is necessary if w is to cross $w = 1$ again. For black hole solutions, the argument is dependent upon the value of $w(r_h)$. If $w(r_h) = 1$, then $w \equiv 1$ for all r, and the black hole geometry is Schwarzschild-adS. If $w(r_h) > 1$, then initially w is increasing, and the same argument used above for the regular solutions applies. If $w(r_h) < 1$, then initially w is decreasing, and once again w cannot subsequently increase to cross $w = 1$. Exactly the same argument works in both cases if w is negative everywhere rather than positive.

In addition to the monopole solutions described in this section, dyonic soliton and black hole solutions also exist [7], again in contrast to the situation in asymptotically flat space, when the electric charge must be zero [12]. Due to the non-vanishing electric field, the stability analysis of these solutions is more complex (see, for example, [15] for the stability analysis of hairy black holes with non-vanishing electric field), and in this paper we shall focus only on the monopole solutions. However, we would anticipate that those dyonic solutions for which w has no nodes would also be stable in the odd-parity sector, at least when the electric field is small enough.

3 The pulsation equations

We therefore consider linear fluctuations of a static and purely magnetic soliton or hairy black hole solution in Einstein-Yang-Mills theory with a cosmological constant. In a recent letter [11], we have shown that, at least in the pure EYM case, such fluctuations can be described by a symmetric wave equation for the linearized extrinsic curvature and the linearized electric field. In this section, we review the results obtained there, and generalize to include a cosmological constant. We discuss only the main steps in the derivation of a symmetric and hyperbolic formulation for a static background and refer to [16] for more details.

One starts with the ADM equations of EYMΛ theory, where the metric and the gauge potential assume the form

$$\begin{aligned} \boldsymbol{g} &= -\alpha^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \\ A &= -\Phi\, \alpha dt + \bar{A}_i(dx^i + \beta^i dt). \end{aligned}$$

(Φ and $\bar{A}_i$ are both Lie algebra valued.) On the background, the slicing Σ_t is adapted to the staticity, i.e. the shift β and the time derivative of the 3-metric, $\dot{\bar{g}}_{ij}$ vanish. As a consequence, the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2\alpha}\left(\dot{\bar{g}}_{ij} - L_\beta \bar{g}_{ij}\right) \tag{2}$$

vanishes on the background. Similarly, the electric YM field, defined in terms of the field strength $F = dA + A\wedge A$ and the future pointing normal unit vector field orthogonal to the slices, $n = (\partial_t - \beta)/\alpha$, by

$$E_i = F_{i\mu} n^\mu, \tag{3}$$

is zero for static and purely magnetic solutions. Since K_{ij} and E_i vanish on the background, the tensors δK_{ij} and δE_i, describing linear fluctuations, are invariant with respect to both infinitesimal diffeomorphisms within the slices and infinitesimal gauge transformation of the gauge potential. Hence, it is natural to look for a wave equation in terms of these "vector-invariant" quantities. In order to get such an equation, one then differentiates some evolution equations with respect to the time coordinate t and uses the linearized version of equations (2) and (3) in order to eliminate time derivatives of $\delta\bar{g}_{ij}$ and the gauge potential $\delta\bar{A}_i$. Next, one uses the linearized momentum and Gauss constraint equations and spatial derivatives thereof in order to make the spatial operator both elliptic and (formally) self-adjoint (the relevant scalar product is given below). Finally, one makes use of the freedom in choosing a (space-dependent) reparametrization of the time coordinate in order to impose the maximal slicing condition,

$$\delta K^i{}_i = 0.$$

As a result, a hyperbolic and symmetric wave equation is obtained from the following combinations of the field equations,

$$\begin{aligned} \Lambda_{ij} &\equiv \frac{\alpha}{\sqrt{-g}}\,\partial_t\,\delta\left(\sqrt{-g}\,E_{ij}\right) - \frac{2}{\alpha}\bar{\nabla}_{(i}\left(\alpha^3\delta E_{j)0}\right) + \frac{1}{\alpha^2}\bar{g}_{ij}\bar{\nabla}^k\left(\alpha^4\delta E_{k0}\right), \\ \Lambda_i^{(YM)} &\equiv -\alpha\partial_t\delta(D^\mu F_{i\mu}) + \frac{1}{\alpha}\bar{D}_i\alpha^3\delta(D^\mu F_{0\mu}) + 2\alpha^2\bar{F}^k{}_i\delta E_{k0}, \end{aligned}$$

where $E_{\mu\nu} = G_{\mu\nu} - 8\pi G T_{\mu\nu} = G_{\mu\nu} - 8\pi G T_{\mu\nu}^{(YM)} + g_{\mu\nu}\Lambda$. Here and in the following, all quantities with a bar refer to the background 3-metric $\bar{g}_{ij}$ and the background magnetic potential $\bar{A}_i$. Latin indices are raised and lowered with $\bar{g}_{ij}$, while the index zero refers to the unit normal n. Finally, $D = d + [A|\,.\,]$ denotes the covariant derivative with respect to the gauge potential.

The wave equation reads

$$\begin{aligned} 0 &= \hat{\Lambda}_{ij} = \hat{\Lambda}_{ij}^{(vac)} + \hat{\Lambda}_{ij}^{(mat)}, \\ 0 &= \Lambda_i^{(YM)}, \end{aligned} \tag{4}$$

where $\hat{\Lambda}_{ij}$ denotes the trace-less part of Λ_{ij}. In terms of the vector-invariant quantities

$$L_{ij} \equiv \alpha\delta K_{ij}, \quad A \equiv \delta\dot{\alpha}, \quad \mathcal{E}_i \equiv \alpha\delta E_i\,,$$

one finds, after using the background equations $E_{\mu\nu} = 0$ and $D^\mu F_{\nu\mu} = 0$,

$$\begin{aligned} \hat{\Lambda}_{ij}^{(vac)} &= \Box_L L_{ij} + 4\bar{\nabla}_{(i}\left(\alpha^k L_{j)k}\right) - 4\alpha_{(i}\bar{\nabla}^k L_{j)k} - 2\alpha\bar{\nabla}^k\left(\frac{\alpha_{(i}}{\alpha}\right)L_{j)k} \\ &\quad -\frac{1}{\alpha}\bar{\nabla}_{(i}\alpha^2\bar{\nabla}_{j)}\left(\frac{A}{\alpha}\right) + \frac{1}{3}\bar{g}_{ij}\left(-\frac{2}{\alpha}\bar{\nabla}^k(\alpha\alpha^l)L_{kl} + \bar{\Delta}A - R_{00}A\right), \\ \frac{\hat{\Lambda}_{ij}^{(mat)}}{4G} &= \alpha\mathrm{Tr}\left\{\bar{F}^k{}_i\bar{F}^l{}_j L_{kl} + \frac{1}{4}\bar{F}_{kl}\bar{F}^{kl}L_{ij} - \frac{1}{3}\bar{g}_{ij}\bar{F}^{ks}\bar{F}^l{}_s L_{kl}\right\} - 8G\alpha\Lambda L_{ij} \\ &\quad -\mathrm{Tr}\left\{\bar{D}_k\left(\alpha\bar{F}_{(i}{}^k\mathcal{E}_{j)}\right) + \frac{\mathcal{E}_k}{\alpha}\bar{D}_{(i}\left(\alpha^2\bar{F}_{j)}{}^k\right) + \frac{\alpha^2}{3}\bar{g}_{ij}\bar{D}_k\left(\frac{\mathcal{E}^l}{\alpha}\right)\bar{F}^{kl}\right\}, \\ \Lambda_i^{(YM)} &= \Box_E\mathcal{E}_i + 4G\alpha\mathrm{Tr}\left(\bar{F}^{lj}\mathcal{E}_l\right)\bar{F}_{ij} \\ &\quad +2\alpha\bar{F}^{jk}\bar{\nabla}_k L_{ij} - \frac{2}{\alpha}L_{kj}\bar{D}^j(\alpha^2\bar{F}^{ki}) - \alpha\bar{F}_{ij}\bar{\nabla}^j\left(\frac{A}{\alpha}\right), \end{aligned}$$

where Tr stands for an Ad-invariant scalar product on the Lie algebra and where all second partial derivatives are contained in the hyperbolic operators $\Box_L$ and $\Box_E$, defined by

$$\begin{aligned} \Box_L L_{ij} &= \left(\frac{1}{\alpha}\partial_t^2 - \bar{\nabla}^k\alpha\bar{\nabla}_k\right)L_{ij} + 2\alpha\bar{R}^k{}_{(i}L_{j)k} - 2\alpha\bar{R}_{kilj}L^{kl}, \\ \Box_E\mathcal{E}_i &= \frac{1}{\alpha}\ddot{\mathcal{E}}_i + 2\bar{D}^j\left(\alpha\bar{D}_{[i}\mathcal{E}_{j]}\right) - \frac{1}{\alpha}\bar{D}_i\alpha^3\bar{D}^j\left(\frac{\mathcal{E}_j}{\alpha}\right) - \alpha[\bar{F}_{ij},\mathcal{E}^j]. \end{aligned}$$

Note that the spatial parts of the operators defined in (4) are symmetric with respect to the scalar product

$$\langle (L^{(1)}, \mathcal{E}^{(1)}) \, , \, (L^{(2)}, \mathcal{E}^{(2)}) \rangle \\ \equiv \int\limits_{\Sigma} \left[\bar{g}^{ik} \bar{g}^{jl} L^{(1)}_{ij} L^{(2)}_{kl} + 2G \mathrm{Tr} \left\{ \bar{g}^{ij} \mathcal{E}^{(1)}_i \mathcal{E}^{(2)}_j \right\} \right] \sqrt{\bar{g}} \, d^3x. \tag{5}$$

The constraint equations are the linearized momentum constraint,

$$0 = \alpha \delta E_{i0} = \alpha \bar{\nabla}^j \left(\frac{L_{ij}}{\alpha} \right) - 2G \mathrm{Tr} \left(\bar{F}^j_i \mathcal{E}_j \right), \tag{6}$$

and the linearized Gauss constraint

$$0 = -\alpha \delta (D^\mu F_{0\mu}) = \alpha \bar{D}^j \left(\frac{\mathcal{E}_j}{\alpha} \right).$$

Additional constraints involving also perturbations of the metric and the gauge potential themselves are the Hamiltonian constraint and all evolution equations, which we had differentiated in time in order to construct the wave operator.

Since we adopt the maximal slicing condition, there is an elliptic equation for A, which is obtained from the trace of the tensor Λ_{ij} ,

$$\left(\bar{\Delta} - R_{00} \right) A = 2 \bar{\nabla}^k \left\{ \alpha^l L_{kl} + G \mathrm{Tr} (\alpha \bar{F}_{lk} \mathcal{E}^l) \right\} - 2G\alpha \mathrm{Tr} \left\{ \bar{F}^{km} \bar{F}^l_{\ m} L_{kl} \right\}, \tag{7}$$

where the momentum constraint equation (6) and the background equations have been used in order to simplify the equation. Here R_{00} is the 00-component of the Ricci tensor. From the background equations, one finds that $R_{00} = G\mathrm{Tr}(\bar{F}^{kl}\bar{F}_{kl})/2 - \Lambda$, and therefore, for a negative Λ, the operator on the left-hand side of equation (7) is negative and the equation is solvable. As we have argued in [16], this equation decouples when one projects the wave operator onto the momentum constraint manifold. In the case of odd-parity perturbations on a spherical symmetric background, as we are going to study below, equation (7) is in fact void since it is a scalar equation.

4 Odd-parity fluctuations

4.1 The form of the perturbation equations

We now apply the curvature-based perturbation theory in order to study the linear stability of the solutions discussed in section 2. Since the background is spherically symmetric in this case, it is convenient to expand the linearized extrinsic curvature and electric YM field in terms of spherical tensor harmonics. Perturbations belonging to different choices of the angular momentum numbers

ℓ and m decouple. Furthermore, the tensor harmonics can be divided into parities. Here, we consider only odd-parity (axial) perturbations. The even-parity (polar) case will be subject to a future article. Since the perturbations of all scalar quantities vanish for odd-parity perturbations, the variation of the lapse and the trace of the extrinsic curvature do not appear and therefore the amplitudes parametrizing δK_{ij} and δE_i are fully coordinate- and gauge-invariant.

How to expand L_{ij} and $\mathcal{E}_i$ in spherical tensor harmonics and how to find the corresponding expansion of the wave operator is explained in Appendix A. We introduce a coordinate ρ such that $\partial_\rho = NS\partial_r$. Since we are dealing with geometries which are asymptotically adS, ρ does not tend to infinity as $r \to \infty$, but instead ρ tends to a finite value ρ_{max}. We shall use the notation ρ_0 to denote the lowest value of ρ, which is 0 at the origin for solitonic solutions, and $-\infty$ at the event horizon for black holes. Using this "tortoise" coordinate, the resulting wave equation is

$$\left(\partial_t^2 - \partial_\rho^2 + \boldsymbol{A}\partial_\rho + \partial_\rho \boldsymbol{A} + \boldsymbol{S}\right) U = 0, \tag{8}$$

where $U \equiv (h, a, b, k, c, d, e)$, with h and k parametrizing metric perturbations and a, b, c, d and e parametrizing the YM field. The matrices $\boldsymbol{A}$ and $\boldsymbol{S}$ are antisymmetric and symmetric, respectively, and have the block structure

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & 0 \\ 0 & \boldsymbol{A}_2 \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_t^T \\ \boldsymbol{S}_t & \boldsymbol{S}_2 \end{pmatrix},$$

where $\boldsymbol{A}_1$ and $\boldsymbol{S}_1$ are 3×3 matrices, whereas $\boldsymbol{A}_2$ and $\boldsymbol{S}_2$ are 4×4 matrices. Defining

$$\gamma \equiv \frac{\alpha}{r}, \quad f \equiv w^2 + 1, \quad u \equiv \frac{w_\rho}{r}, \quad v \equiv 2\sqrt{G}\,\frac{w^2-1}{r},$$

the non-vanishing matrix elements of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ can be written as

$$\begin{aligned} (\boldsymbol{A}_1)_{13} &= -(\boldsymbol{A}_1)_{31} = -\sqrt{G}\,u, \\ (\boldsymbol{A}_2)_{14} &= -(\boldsymbol{A}_2)_{41} = -\sqrt{2G}\,u. \end{aligned}$$

The matrix $\boldsymbol{S}$ is given by

$$\boldsymbol{S}_1 = \gamma^2 \begin{pmatrix} \frac{r}{\gamma^3}\left(\frac{\gamma}{r}\right)_{\rho\rho} + \lambda + v^2 & sym. & sym. \\ \mu\, v & \frac{\gamma_{\rho\rho}}{\gamma^3} + [\lambda + 2f] & sym. \\ -w\,v - \frac{\sqrt{G}\,r^2}{\gamma^4}\left(\frac{\gamma^2 u}{r^2}\right)_\rho & -2\mu w & \frac{\gamma_{\rho\rho}}{\gamma^3} + [\lambda + f] + \frac{4Gu^2}{\gamma^2} \end{pmatrix},$$

$$\boldsymbol{S}_2 = \gamma^2 \begin{pmatrix} \frac{r_{\rho\rho}}{r\gamma^2} + \lambda + \frac{4Gu^2}{\gamma^2} & sym. & sym. & sym. \\ \sqrt{\lambda}\,v & [\lambda + 2f] + v^2 & sym. & sym. \\ 0 & \sqrt{2}\mu w & [\lambda - 2 + 3f] & sym. \\ \sqrt{2G}\frac{u_\rho}{\gamma^2} & -\sqrt{2\lambda}\,w & 0 & [\mu^2 - f] \end{pmatrix},$$

$$
\boldsymbol{S}_t = \left(\begin{array}{ccc} 2\sqrt{\lambda}\,\gamma_\rho & 0 & 0 \\ \frac{r}{\gamma}\left[\frac{\gamma^2}{r}v\right]_\rho + 2\sqrt{G}\,\gamma w\,u & 2\mu\gamma_\rho & -2(\gamma w)_\rho - 2\sqrt{G}\gamma\,uv \\ -\sqrt{2G}\,\mu\gamma\,u & 2\sqrt{2}(\gamma w)_\rho & -\sqrt{2}\mu\gamma_\rho \\ -\sqrt{2G\lambda}\,\gamma\,u & 0 & \sqrt{2\lambda}\gamma_\rho \end{array} \right),
$$

where $\mu^2 = \ell(\ell+1)$ and $\lambda = \mu^2 - 2$ depend on the angular momentum number ℓ. The constraint equations are found to be

$$
\begin{aligned}
0 &= \frac{\gamma}{r}\,\partial_\rho\left(\frac{r}{\gamma}\,h\right) + 2\sqrt{G}u\,b - \sqrt{\lambda}\gamma k - \gamma\,v\,c, \\
0 &= \gamma\,\partial_\rho\left(\frac{1}{\gamma}\,a\right) - \mu\gamma\,c - \sqrt{2}\gamma w\,d, \\
0 &= \gamma\,\partial_\rho\left(\frac{1}{\gamma}\,b\right) + \gamma w\,c + \frac{\mu}{\sqrt{2}}\gamma\,d - \sqrt{\frac{\lambda}{2}}\,\gamma\,e. \qquad (9)
\end{aligned}
$$

How to solve the initial value problem in terms of gauge-invariant quantities is discussed in Appendix B. A nice fact is that the cosmological constant does not appear explicitly in the above system, it appears only via the background quantities S, N and w. As a consequence, the stability can be discussed on the same lines as for asymptotically flat solutions [9]. Below, we study perturbations which are smooth and which vanish at the boundary points $\rho = \rho_0$ ($r = 0$ or $r = r_h$ as applicable) and $\rho = \rho_{max}$ ($r = \infty$). The spatial part of the wave operator defined in (8) is symmetric for such perturbations, and since the operator is real, self-adjoint extensions exist. The spectrum of the self-adjoint extension which corresponds to physical boundary conditions decides the linear stability of the background solution. It is, however, beyond the scope of this article to determine the self-adjoint extensions. We will rather base our stability argument on an energy estimate.

4.2 The projection onto the constraint manifold and stability

In order to study the linear stability of the background solutions, one has to consider the time evolution of perturbations U, which are restricted to the constraint manifold defined by (9). Our next aim is therefore to separate the purely dynamical degrees of freedom from the constraint violating modes. Generalizing ideas introduced by Anderson *et al.* [17], we define the constraint variables u_c by the right-hand side of (9). We are then looking for a dynamical variable, u_p, say, such that the wave equation assumes the form

$$
\left[\partial_t^2 - \partial_\rho^2 + \left(\begin{array}{cc} \boldsymbol{V}_c & 0 \\ \boldsymbol{V}_{pc} & \boldsymbol{V}_p \end{array} \right)\right] \left(\begin{array}{c} u_c \\ u_p \end{array} \right) = 0, \qquad (10)
$$

when expressed in the new variables (u_c, u_p). The operators $\boldsymbol{V}_c$, $\boldsymbol{V}_{pc}$ and $\boldsymbol{V}_p$ are permitted to have up to first order spatial derivatives only. The components

in $\boldsymbol{V}_{cp}$ have to vanish in order for the constraint variables to evolve. On the constraint manifold, $u_c = 0$, we then get

$$\left[\partial_t^2 - \partial_\rho^2 + \boldsymbol{V}_p\right] u_p = 0,$$

which is a wave equation for the dynamical variables u_p. Of course, it would be nice if the transformation can be found in such a way that the new potential $\boldsymbol{V}_p$ is automatically symmetric. In our case, it turns out that this is possible.

As a simple example we first decouple the dynamical modes from the constraint violating ones in the vacuum case [16], where the YM amplitudes a, b, c, d and e vanish, and where $w = 1$. The idea is to find a first order linear transformation of the form

$$\begin{pmatrix} u_c \\ u_p \end{pmatrix} = \boldsymbol{B} \begin{pmatrix} h \\ k \end{pmatrix}, \quad \boldsymbol{B} = \partial_\rho + \boldsymbol{C},$$

such that the spatial part of the wave operator factorizes,

$$-\partial_\rho^2 + \begin{pmatrix} \frac{r}{\gamma}\left(\frac{\gamma}{r}\right)_{\rho\rho} + \lambda\gamma^2 & 2\sqrt{\lambda}\gamma_\rho \\ 2\sqrt{\lambda}\gamma_\rho & \frac{r_{\rho\rho}}{r} + \lambda\gamma^2 \end{pmatrix} = \boldsymbol{B}^\dagger \boldsymbol{B}, \tag{11}$$

where $\boldsymbol{B}^\dagger = -\partial_\rho + \boldsymbol{C}^T$ is the (formal) adjoint of $\boldsymbol{B}$. The desired wave equation (10) is then obtained upon applying the operator $\boldsymbol{B}$ to the left of the original wave equation (8). This yields

$$\left[\partial_t^2 + \boldsymbol{B}\boldsymbol{B}^\dagger\right] \begin{pmatrix} u_c \\ u_p \end{pmatrix} = 0.$$

Since $\boldsymbol{B}\boldsymbol{B}^\dagger$ is symmetric, the fact that $\boldsymbol{V}_{cp}$ must vanish implies that $\boldsymbol{V}_{pc}$ has to vanish as well. Provided that the factorization (11) can be found, the constraint and dynamical variables can therefore be decoupled completely.

It is far from obvious that such a transformation exists, since equation (11) automatically implies that the spatial operator is positive. Nevertheless, it turns out that the factorization can be found in our case. Indeed, by making the ansatz

$$\boldsymbol{B} = \partial_\rho + \begin{pmatrix} \frac{\gamma}{r}\left(\frac{r}{\gamma}\right)_\rho & -\sqrt{\lambda}\gamma \\ -\sqrt{\lambda}\gamma & A \end{pmatrix},$$

where the first row of the matrix has been chosen such that u_c is defined by the right-hand side of the first equation in (9) and the second row such that there are no first order derivatives in $\boldsymbol{B}^\dagger\boldsymbol{B}$, equation (11) is fulfilled if $A = -\frac{r_\rho}{r}$. The transformed spatial operator is

$$\boldsymbol{B}\boldsymbol{B}^\dagger = -\partial_\rho^2 + \begin{pmatrix} \frac{\gamma}{r}\left(\frac{r}{\gamma}\right)_{\rho\rho} + \gamma^2\lambda & 0 \\ 0 & r\left(\frac{1}{r}\right)_{\rho\rho} + \gamma^2\lambda \end{pmatrix}.$$

Using $r(1/r)_{\rho\rho} = \gamma^2(2 - 6m/r)$ for a Schwarzschild background, we recognize that the dynamical variables are governed by the Regge-Wheeler equation [18]. Since the spatial operator has the form $\boldsymbol{B}^\dagger \boldsymbol{B}$, it is positive on the space of smooth perturbations with compact support. Therefore, in the odd-parity sector, the linear stability of the Schwarzschild black hole can be established without needing the explicit form of the Regge-Wheeler potential. This sort of topological argument will be important below, since the background solutions in EYM theory are not known in closed form.

We now show how to decouple the constraint and dynamical variables for the full EYM wave equation (8). Here, the ansatz is

$$\begin{pmatrix} u_c \\ u_p \end{pmatrix} = \boldsymbol{B} U, \quad \boldsymbol{B} = \partial_\rho - \boldsymbol{A} + \boldsymbol{C}_S \,,$$

where the antisymmetric matrix $\boldsymbol{A}$ is given above and is introduced in order to reproduce the first order derivatives in the wave operator in (8). The matrix $\boldsymbol{C}_S$ is assumed to be symmetric. Writing this matrix in block form,

$$\boldsymbol{C}_S = \begin{pmatrix} \boldsymbol{C}_1 + \boldsymbol{A}_1 & \boldsymbol{C}_t^T \\ \boldsymbol{C}_t & \boldsymbol{C}_2 \end{pmatrix},$$

we must have

$$\begin{aligned} \boldsymbol{C}_1 &= \begin{pmatrix} \frac{\gamma}{r}\left(\frac{r}{\gamma}\right)_\rho & 0 & 2\sqrt{G}\,u \\ 0 & -\frac{\gamma_\rho}{\gamma} & 0 \\ 0 & 0 & -\frac{\gamma_\rho}{\gamma} \end{pmatrix}, \\ \boldsymbol{C}_t^T &= \gamma \begin{pmatrix} -\sqrt{\lambda} & -v & 0 & 0 \\ 0 & -\mu & -\sqrt{2}w & 0 \\ 0 & w & \frac{\mu}{\sqrt{2}} & -\sqrt{\frac{\lambda}{2}} \end{pmatrix}, \end{aligned}$$

in order for the constraint variables to be defined as the right-hand side of (9). Hence, only the symmetric 4×4 matrix $\boldsymbol{C}_2$ has to be matched.

Equating $\boldsymbol{B}^\dagger \boldsymbol{B}$ with the spatial part of the wave operator, $-\partial_\rho^2 + \boldsymbol{A}\partial_\rho + \partial_\rho \boldsymbol{A} + \boldsymbol{S}$ yields the following equations. First, we find the equation

$$-\partial_\rho \boldsymbol{C}_1 + \boldsymbol{C}_1^2 + \boldsymbol{C}_t^T \boldsymbol{C}_t + 2\boldsymbol{A}_1 \boldsymbol{C}_1 = \boldsymbol{S}_1 + \partial_\rho \boldsymbol{A}_1 \,,$$

which is a consistency condition that can be shown to hold automatically. Next, we have a linear algebraic equation for the unknown, symmetric matrix $\boldsymbol{C}_2$,

$$\boldsymbol{C}_t^T \boldsymbol{C}_2 = \boldsymbol{Q}^T, \tag{12}$$

where $\boldsymbol{Q} \equiv \boldsymbol{S}_t + \partial_\rho \boldsymbol{C}_t - \boldsymbol{C}_t \boldsymbol{C}_1 - \boldsymbol{A}_2 \boldsymbol{C}_t$. Explicitly, we find

$$\boldsymbol{Q}^T = \gamma \begin{pmatrix} \sqrt{\lambda}\,\frac{r_\rho}{r} & 2\sqrt{G}\,wu & -\sqrt{2G}\mu u & 0 \\ 0 & 0 & \sqrt{2}\,w_\rho & 0 \\ \sqrt{G\lambda}\,u & -w_\rho & 0 & 0 \end{pmatrix}.$$

Finally, we get a differential equation for $\boldsymbol{C}_2$:

$$- \partial_\rho \boldsymbol{C}_2 + \boldsymbol{C}_2^2 + [\boldsymbol{A}_2, \boldsymbol{C}_2] = \boldsymbol{T}, \tag{13}$$

where $\boldsymbol{T} \equiv \boldsymbol{S}_2 - \boldsymbol{C}_t \boldsymbol{C}_t^T + \boldsymbol{A}_2^2$, so that

$$\boldsymbol{T} = \begin{pmatrix} \frac{r_{\rho\rho}}{r} + 2Gu^2 & sym. & sym. & sym. \\ 0 & \gamma^2 w^2 & sym. & sym. \\ 0 & -\frac{\mu}{\sqrt{2}}\gamma^2 w & \gamma^2[\frac{\lambda}{2} + w^2] & sym. \\ \sqrt{2G}\, u_\rho & -\sqrt{\frac{\lambda}{2}}\,\gamma^2 w & \frac{1}{2}\mu\sqrt{\lambda}\,\gamma^2 & \gamma^2[\frac{\mu^2}{2} - w^2] - 2Gu^2 \end{pmatrix}.$$

Equation (13) can be shown to be consistent with the linear equation (12). More precisely, if we set $\mathcal{C} \equiv \boldsymbol{C}_2 \boldsymbol{C}_t^T - \boldsymbol{Q}$, we can show that the differential equation for $\boldsymbol{C}_2$ implies that

$$-\partial_\rho \mathcal{C} + (\boldsymbol{C}_2 + \boldsymbol{A}_2)\mathcal{C} - \mathcal{C}(2\boldsymbol{A}_1 + \boldsymbol{C}_1) = 0.$$

Suppose that we found a solution to the equations (12) and (13). Then, the wave equation (8) is equivalent to the wave equation

$$\left(\partial_t^2 + \boldsymbol{B}\boldsymbol{B}^\dagger\right) V = \left(\partial_t^2 - \partial_\rho^2 + \boldsymbol{A}\partial_\rho + \partial_\rho \boldsymbol{A} + \tilde{\boldsymbol{S}}\right) V = 0,$$

where $V = \boldsymbol{B}U$. The symmetric potential $\tilde{\boldsymbol{S}}$ is given in block form by

$$\begin{aligned} \tilde{\boldsymbol{S}}_1 &= \boldsymbol{S}_1 + 2[\boldsymbol{C}_1, \boldsymbol{A}_1] + 2\partial_\rho(\boldsymbol{C}_1 + \boldsymbol{A}_1), \\ \tilde{\boldsymbol{S}}_t &= \boldsymbol{S}_t + 2(\boldsymbol{C}_t \boldsymbol{A}_1 - \boldsymbol{A}_2 \boldsymbol{C}_t) + 2\partial_\rho \boldsymbol{C}_t, \\ \tilde{\boldsymbol{S}}_2 &= \boldsymbol{S}_2 + 2[\boldsymbol{C}_2, \boldsymbol{A}_2] + 2\partial_\rho \boldsymbol{C}_2. \end{aligned}$$

Since $\tilde{\boldsymbol{S}}_1$ and $\tilde{\boldsymbol{S}}_t$ do not depend on $\boldsymbol{C}_2$, they can be computed directly. One obtains

$$\frac{\tilde{\boldsymbol{S}}_1}{\gamma^2} = \begin{pmatrix} \frac{1}{r\gamma}\left(\frac{r}{\gamma}\right)_{\rho\rho} + \lambda + v^2 + \frac{4Gu^2}{\gamma^2} & sym. & sym. \\ \mu v & \frac{1}{\gamma}\left(\frac{1}{\gamma}\right)_{\rho\rho} + \lambda + 2f & sym. \\ -w\, v + \sqrt{G}\left(\frac{u}{\gamma^2}\right)_\rho & -2\mu w & \frac{1}{\gamma}\left(\frac{1}{\gamma}\right)_{\rho\rho} + \lambda + f \end{pmatrix},$$

and $\tilde{\boldsymbol{S}}_t = 0$. The fact that $\tilde{\boldsymbol{S}}_t$ vanishes shows that - provided that we can solve the equation (13) - the constraint variable and the dynamical variables decouple from each other and are both governed by a symmetric wave equation. The evolution equation for the constraint variables is given by the symmetric wave equation

$$\left(\partial_t^2 - \partial_\rho^2 + \boldsymbol{A}_1\partial_\rho + \partial_\rho \boldsymbol{A}_1 + \tilde{\boldsymbol{S}}_1\right) u_c = 0,$$

and therefore, initial data which satisfies $u_c = \dot{u}_c = 0$ will satisfy the constraint equations for all later times.

If we have been able to factorize the spatial operator, stability of our solutions is then automatic. More precisely, one can show that for smooth perturbations with compact support, the energy expression

$$E = \frac{1}{2} \int_{\rho_0}^{\rho_{max}} \Big\{ (\dot{U}, \dot{U}) + (\boldsymbol{B}U, \boldsymbol{B}U) \Big\} d\rho,$$

is constant in time. Our boundary conditions are that all perturbations vanish at the origin (or event horizon) and at infinity. However, at this point, one can also require less restrictive boundary conditions. At the horizon, for example, it is sufficient to require that U and $\dot{U}$ are finite. At infinity ($\rho = \rho_{max}$), one can impose the outgoing wave condition $\dot{U} = -\boldsymbol{B}U$, which means that there is no radiation coming from infinity. This condition implies that the energy cannot increase in time. Since $E \geq 0$, the kinetic energy must therefore be bounded and exponentially growing modes cannot exist. The remarks in Appendix B then show that the metric and the gauge potential themselves cannot grow exponentially.

4.3 Factorization of the wave operator

The result of the previous section is that we have shown the stability of our solutions once we have found a factorization of the spatial operator, that is, a solution of the equations (12) and (13). This is the subject of the present section.

For the moment, we exclude the Schwarzscild-adS and the RN-adS backgrounds from the analysis below. These cases have to be treated separately. We start with the distinguished cases $\ell = 0$ and $\ell = 1$. For $\ell = 0$, only the YM amplitudes a and d are present. The linear equation (12) then reduces to $-\sqrt{2}\, w \boldsymbol{C}_2^{(\ell=0)} = \sqrt{2}\, w_\rho$, with the unique solution $\boldsymbol{C}_2^{(\ell=0)} = -w_\rho / w$. Equation (13) is also fulfilled by this choice. Since we are interested in the fundamental solutions where w has no zeros, $\boldsymbol{C}_2^{(\ell=0)}$ is regular, and the transformed potential turns out to be

$$\tilde{\boldsymbol{S}}_2^{(\ell=0)} = \gamma^2 [w^2 + 1] + 2 \left(\frac{w_\rho}{w} \right)^2 ,$$

which is the same as the one obtained in [8] in the sphaleronic sector. The potential $\tilde{\boldsymbol{S}}_2^{(\ell=0)}$ is positive, and has the asymptotic behaviour

$$\begin{aligned}
\tilde{\boldsymbol{S}}_2^{(\ell=0)} &\longrightarrow \frac{2}{r^2} \quad \text{as} \quad r \longrightarrow 0, \\
\tilde{\boldsymbol{S}}_2^{(\ell=0)} &\longrightarrow 0 \quad \text{as} \quad r \longrightarrow r_h, \\
\tilde{\boldsymbol{S}}_2^{(\ell=0)} &\longrightarrow -\frac{\Lambda}{3}(1 + w_\infty^2) \quad \text{as} \quad r \longrightarrow \infty,
\end{aligned}$$

the behaviour at $r = 0$ being the same as for p-waves. Since $\tilde{\boldsymbol{S}}_2^{(\ell=0)}$ is positive, linear stability follows, as discussed at the end of the previous subsection.

[Note that for solutions where w has zeros, $\boldsymbol{C}_2^{(\ell=0)}$ is singular. In this case, the method presented in [19] can be generalized and the existence of exactly n unstable modes established (n being the number of nodes of the function w).]

For $\ell = 1$ the amplitudes k and e are absent. The corresponding rows and columns in the matrices $\boldsymbol{C}_t$, $\boldsymbol{C}_2$, $\boldsymbol{S}_t$, $\boldsymbol{S}_2$ and $\boldsymbol{A}_2$ have therefore to be removed. [The consistency conditions checked so far remain valid, since for $\ell = 1$, the entries in the removed rows and columns decouple from the entries in the other rows and columns.] The linear equation (12) turns out to have the unique solution

$$\boldsymbol{C}_2^{(\ell=1)} = \frac{w_\rho}{1-w^2}\begin{pmatrix} w & -1 \\ -1 & w \end{pmatrix},$$

which also satisfies the differential equation (13). Thus, the constraint and dynamical variables decouple, and the dynamical variables are governed by the following symmetric wave equation:

$$\left(\partial_t^2 - \partial_\rho^2 + \tilde{\boldsymbol{S}}_2^{(\ell=1)}\right) u_p = 0,$$

for

$$u_p = \left(\partial_\rho + \boldsymbol{C}_2^{(\ell=1)}\right)\begin{pmatrix} c \\ e \end{pmatrix} + \gamma \begin{pmatrix} -2\sqrt{G}\,\frac{w^2-1}{r} & -\sqrt{2} & w \\ 0 & -\sqrt{2}\,w & 1 \end{pmatrix}\begin{pmatrix} h \\ a \\ b \end{pmatrix},$$

where the symmetric potential is given by

$$\tilde{\boldsymbol{S}}_2^{(\ell=1)} = \gamma^2 \begin{pmatrix} 2 + 4G\frac{(w^2-1)^2}{r^2} & 4w \\ 4w & 1+w^2 \end{pmatrix} + \frac{2w_\rho^2}{(1-w^2)^2}\begin{pmatrix} 1+w^2 & -2w \\ -2w & 1+w^2 \end{pmatrix}.$$

For solutions in which w has no zeros, the supersymmetric transformation is regular provided that $w \neq \pm 1$. However, as discussed in section 2, for solitonic solutions $w = \pm 1$ only at the origin, and for black holes, w^2 is never unity unless $w \equiv \pm 1$, in which case the geometry is Schwarzschild-adS. The new potential is again positive and has a similar asymptotic behaviour as that for $\ell = 0$. Therefore, the stability follows also in this case.

For solutions in which w does have zeros, the transformation is again regular provided that $w \neq \pm 1$. In asymptotically flat space, w does not cross ± 1 for all regular solitonic and black hole solutions, except that $w = \pm 1$ at the origin for solitons [13]. This is also the case for the majority of solitons and black holes in adS, so we can conclude that these solutions also have no unstable modes for $\ell = 1$. However, there are some regular solutions for which w has zeros and does cross ± 1 away from the origin or event horizon [8], for which we are not able to draw conclusions about the existence of unstable $\ell = 1$ modes.

We now turn to the generic case $\ell \geq 2$: The general solution to the linear equation (12) can be written into the form

$$\boldsymbol{C}_2 = \boldsymbol{D}\left(\boldsymbol{X}_0 + T(\rho)\boldsymbol{X}_1\right)\boldsymbol{D},$$

where $\boldsymbol{D} = \mathrm{diag}(n_2, m_3, m_4, m_5)$ is a positive, constant matrix (the values of n_2, ... , m_5, which are not relevant for what follows, are given in Appendix A). The symmetric matrices $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ are

$$\begin{aligned}
\boldsymbol{X}_0 &= \begin{pmatrix} -2\lambda\mu^2\frac{r_\rho}{r} + 4\sqrt{G} f_1 w\, uv & sym. & sym. & sym. \\ -2\sqrt{2G}\, f_1 w\, u & 2ww_\rho & sym. & sym. \\ 2\sqrt{2G}\, f_1 u & -2w_\rho & 0 & sym. \\ 4\sqrt{2G}\, w^2(1-w^2)u & 2w^2 w_\rho & -2ww_\rho & -f_2 ww_\rho \end{pmatrix}, \\
\boldsymbol{X}_1 &= \begin{pmatrix} 4w^2v^2 & sym. & sym. & sym. \\ -2\sqrt{2G}\, w^2 v & 2w^2 & sym. & sym. \\ 2\sqrt{2G}\, wv & -2w & 2 & sym. \\ \sqrt{2}\, f_2 wv & -f_2 w & f_2 & \frac{1}{2}f_2^2 \end{pmatrix},
\end{aligned}$$

where $f_1 = \lambda + 2w^2$ and $f_2 = \mu^2 - 2w^2$. So far, the function $T(\rho)$ is arbitrary. Introducing this into equation (13) yields the following non-linear first order differential equation for T:

$$-\partial_\rho T + \mathcal{A}T^2 + \mathcal{B}T + \mathcal{C} = 0, \tag{14}$$

with

$$\begin{aligned}
\mu^2\lambda\mathcal{A} &= \frac{8G}{r^2}w^2(1-w^2)^2 + 4\left(w^2 - 1 - \frac{\lambda}{4}\right)^2 + 4\lambda + \frac{7}{4}\lambda^2, \\
\mu^2\lambda\mathcal{B} &= 8\left[\frac{2G}{r^2}(\lambda + 2w^2) + 1\right](w^2-1)ww_\rho\,, \\
\mu^2\lambda\mathcal{C} &= \left[\frac{2G}{r^2}(\lambda + 2w^2) + 2\lambda + 4w^2\right]w_\rho^2 - \mu^2\lambda\left(\frac{\lambda}{2} + w^2\right)\gamma^2. \qquad (15)
\end{aligned}$$

To summarize, it is sufficient to show that the single differential equation (14) admits a global solution with appropriate boundary conditions in order to show the stability of the evolution system (8), which is a wave equation for seven amplitudes. In the next subsection we shall complete the stability proof by showing the existence of a globally regular solution to (14).

Finally, we turn to the stability of the Schwarzschild-adS and RN-adS black holes. The stability of the Schwarzschild-adS metric can be established on the same lines as above, but one has to take into account that for $\ell = 1$, the matrix $\boldsymbol{C}_2$ is no longer uniquely specified by the linear equation (12), and one obtains a differential equation similar to (14). Alternatively, one can also use the perturbation formalism presented in Ref. [20], since in the odd-parity sector, the cosmological constant does not appear explicitly.

While the stability proof goes through for $\ell \geq 1$, the RN-adS solution turns out to be unstable with respect to odd-parity radial perturbations of the non-Abelian part of the YM field which is parametrized by the amplitude d. When $w \equiv 0$, the amplitudes a and d decouple, and d is governed by the equation

$$\left(\partial_t^2 - \partial_\rho^2 - \frac{N}{r^2}\right)d = 0.$$

An easy way to find the instability is to use a trial function [21]. We define a sequence of functions $d(\rho)$ as follows:

$$d(\rho) = \mathcal{Z}\left(\frac{\rho}{J}\right),$$

for $J = 1, 2, \ldots$. The function $\mathcal{Z}(\rho) \in [0,1]$ is defined by [21]

$$\begin{aligned} \mathcal{Z}(\rho) &= 0 \qquad \text{for } \rho < -P-1, \\ \mathcal{Z}(\rho) &= 1 \qquad \text{for } \rho > -P, \end{aligned}$$

and

$$0 \leq \frac{d\mathcal{Z}}{d\rho} \leq Q \qquad \text{for } \rho \in [-P-1, -P],$$

where P and Q are arbitrary positive numbers. The expectation value of the spatial operator yields, after some calculation,

$$\begin{aligned} \left\langle d \left| -\partial_\rho{}^2 - \frac{N}{r^2} \right| d \right\rangle &= \int_{r_h}^{\infty} \left(N d_r^2 - \frac{d^2}{r^2} \right) dr \\ &\leq \frac{Q^2}{J} - \frac{1}{r_h}, \end{aligned}$$

while $\langle d \,|\, d \rangle < \infty$. This expectation value is negative provided we choose J to be sufficiently large. Therefore we conclude that there are exponentially growing modes and the RN-adS black hole is unstable.

4.4 Global solutions to equation (14)

It remains at this stage to show that equation (14) has globally regular solutions. We shall begin by considering only those solitons and black holes for which w has no zeros, since these are the solutions of most interest to us.

In order to see that equation (14) admits global solutions, one rewrites it as a linear, second order differential equation using the transformation

$$T = -\frac{1}{\mathcal{A}} \frac{z_\rho}{z}. \tag{16}$$

Going back to the radial coordinate r, this yields

$$\frac{\partial^2 z}{\partial r^2} + \left[\frac{1}{NS} \frac{\partial (NS)}{\partial r} - \tilde{\mathcal{B}} - \frac{1}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial r} \right] \frac{\partial z}{\partial r} + \mathcal{A}\tilde{\mathcal{C}} z = 0, \tag{17}$$

where

$$\begin{aligned} \mu^2 \lambda \tilde{\mathcal{B}} &= 8 \left[\frac{2G}{r^2}(\lambda + 2w^2) + 1 \right] (w^2 - 1) w w_r, \\ \mu^2 \lambda \tilde{\mathcal{C}} &= \left[\frac{2G}{r^2}(\lambda + 2w^2) + 2\lambda + 4w^2 \right] w_r^2 - \frac{\mu^2 \lambda}{N r^2} \left(\frac{\lambda}{2} + w^2 \right). \end{aligned} \tag{18}$$

The equation (17) is regular everywhere except at the origin, infinity, and (if one exists) the event horizon $r = r_h$, where there are regular singular points. Using the asymptotic expansions and the standard Frobenius method, the solutions for z near each of these points can be found.

Near the origin, for globally regular (solitonic) solutions, the functions in (17) behave as:

$$\begin{aligned}
\mu^2\lambda\mathcal{A} &= 2\lambda^2 + 4\lambda + O(r^2), \\
\mu^2\lambda\tilde{\mathcal{B}} &= O(r^3), \\
\mu^2\lambda\tilde{\mathcal{C}} &= -\frac{\mu^2\lambda}{r^2}\left(\frac{\lambda}{2}+1\right) + O(1),
\end{aligned}$$

so that the origin is a regular singular point, and the two linearly independent solutions have the form

$$z \sim r^{\ell+1}, \quad r^{-\ell}.$$

If, instead, we are considering black hole solutions with an event horizon at $r = r_h$, then $\mathcal{A}$ and $\tilde{\mathcal{B}}$ are regular functions at the event horizon, while $\tilde{\mathcal{C}}$ diverges as $(r - r_h)^{-1}$. Therefore, again there is a regular singular point, this time the indicial equation for z has a repeated root of zero, so that the linearly independent solutions have the form:

$$z = O(1), \quad O(\log(r - r_h)).$$

As expected, the negative cosmological constant means that the analysis at infinity is different to that in the asymptotically flat case [9]. Letting $s = 1/r$, the behaviour of the functions in (17) is:

$$\begin{aligned}
NS &= -\frac{\Lambda}{3s^2} + O(1/s), \\
\mathcal{A} &= A_\infty + A_1 s + O(s^2), \\
\tilde{\mathcal{B}} &= B_\infty s^2 + O(s^3), \\
\tilde{\mathcal{C}} &= C_\infty s^4 + O(s^5),
\end{aligned}$$

which implies that the point $s = 0$ is, in fact, a regular point. As a consequence, the asymptotic forms of the linearly independent solutions for z at infinity are:

$$z = O(1), \quad O(1/r).$$

The proof of the existence of global solutions to equation (17) can now proceed as in the asymptotically flat case [9]. Consider firstly black hole solutions. We start near the event horizon with the solution having $z = O(1)$ (i.e. with no logarithmic term), so that z is positive for sufficiently small $r - r_h$. At a maximum of z, where $dz/dr = 0$, from (17) we have

$$\frac{d^2z}{dr^2} = -\mathcal{A}\tilde{\mathcal{C}}z.$$

It is clear from (15) that $\mathcal{A} > 0$ always, so we need to investigate the sign of $\tilde{\mathcal{C}}$. In the asymptotically flat case [9], it was checked numerically that $\tilde{\mathcal{C}} < 0$, but in our situation, we can prove this analytically for sufficiently large $|\Lambda|$.

First fix $\ell = 2$. It was proved in [8] (see also section 2) that, as $|\Lambda| \to \infty$,

$$N \sim O(|\Lambda|), \qquad \frac{dw}{dr} \sim o\left(|\Lambda|^{\frac{1}{2}}\right)$$

for all r. Hence, for sufficiently large $|\Lambda|$, both terms on the right-hand-side of (18) are vanishing, but the first term is $o(|\Lambda|^{-1})$ and the second term $O(|\Lambda|^{-1})$, so the first term is vanishing more rapidly. Therefore, for sufficiently large $|\Lambda|$, we have that $\tilde{\mathcal{C}} < 0$ for all r, for this particular value of ℓ. If we now replace $\ell \to \ell + 1$, then $\mu^2 \lambda \tilde{\mathcal{C}} \to \mu^2 \lambda \tilde{\mathcal{C}} + \mathcal{F}$, where a lengthy calculation (using the fact that $\tilde{\mathcal{C}} < 0$ for the original value of ℓ) yields

$$\begin{aligned}\mathcal{F} &< \frac{\lambda + 2w^2}{(\ell-1)^2(\ell+2)} \left[-\frac{2G\lambda}{r^2}(2\ell^2 + 6\ell) - 2\lambda(2\ell^2 + 6\ell) \right. \\ &\quad \left. -8w^2(\ell-1)(\ell+2)\left(\frac{G}{r^2} + 1\right)\right] \\ &< 0,\end{aligned}$$

where λ and μ are calculated with the original value of ℓ. Therefore, as ℓ increases $\tilde{\mathcal{C}}$ decreases, and is therefore negative for all values of ℓ, for sufficiently large $|\Lambda|$.

We conclude that z cannot have a maximum if it is positive, nor minimum if it is negative. Therefore, since z is positive close to the event horizon, it will remain strictly positive, and take the form

$$z = z_\infty + O(r^{-1})$$

at infinity, where $z_\infty > 0$. Since $\mathcal{A}$ is positive, and z has no zeros, using the definition (16) we see that T is regular and exists globally with the asymptotic behaviour

$$T \sim (r - r_h)$$

near the event horizon, and

$$T = O(1)$$

at infinity. As an example, the function T/S for a particular black hole solution with $\Lambda = -100$ is shown in figure 1.

For globally regular solutions, a similar argument holds, using the solution $z \sim r^{\ell+1}$ near the origin. In this case the function T behaves like r^{-1} for small r but the behaviour at infinity is unchanged.

A careful analysis reveals that again, the behaviour of the transformed potential $\tilde{\boldsymbol{S}}_2$ is similar to the $\ell = 0$ case: $\tilde{\boldsymbol{S}}_2 = O(1/r^2), O(r - r_h), O(1)$ near $r = 0, r_h, \infty$, respectively.

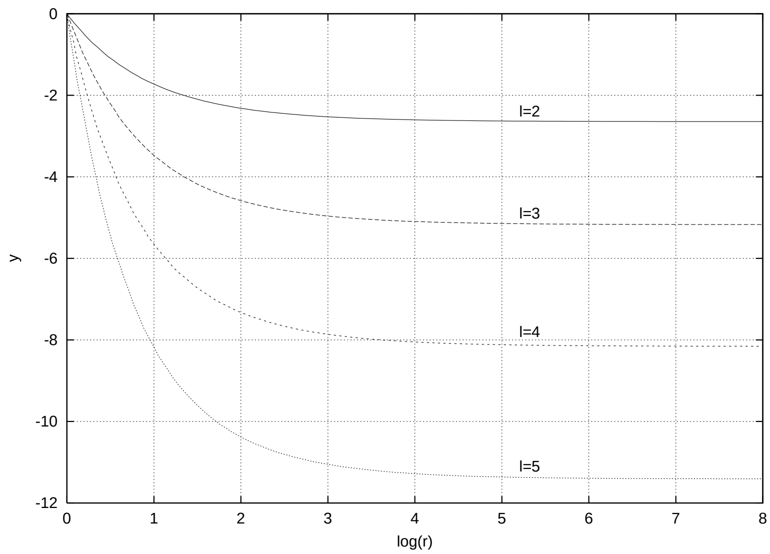

Figure 1: The function $y \equiv T/S$ for $\ell = 2, 3, 4, 5$. The background solution is a black hole with $\Lambda = -100$, $r_h = 1$, and $w(r_h) = 0.9$.

We have now completed the proof that those black holes and solitons which have no unstable modes in the odd-parity sector for spherically symmetric perturbations also have no unstable modes in the odd-parity sector when we consider non-spherically symmetric perturbations. In other words, we have proved the (odd-parity) stability of those solutions in which w has no zeros and $|\Lambda|$ is sufficiently large.

A similar analysis shows that the Bartnik-McKinnon solitons and the corresponding black holes with hair have no unstable modes with $\ell > 0$ and odd parity [9]. The only difference there, is that the background quantities imply that the function T behaves like $1/r$ near infinity. Instead of approaching a constant value, the transformed potential falls off as $1/r^2$ when $r \to \infty$.

The question, therefore, is whether our analysis can be extended to other values of $|\Lambda|$. The answer, surprisingly, is no. The crucial part of the work is the sign of the function $\tilde{\mathcal{C}}$. For large $|\Lambda|$ we have been able to show analytically that $\tilde{\mathcal{C}}$ is always negative. When $\Lambda = 0$, numerical analysis showed this to also be the case [9]. However, for small, negative Λ, we have found numerically that $\tilde{\mathcal{C}}$ can be positive and the solutions to (14) blows up (for example, for black hole solutions with $\Lambda = -0.001$, $r_h = 1$ and $w(r_h) = 0.642$). Therefore our analysis does not extend and it may be that these solutions have additional instabilities (they are already unstable for spherically symmetric perturbations). In order to find such instabilities, one could use the nodal theorem for coupled, symmetric systems of Schrödinger equations proved in Ref. [22] to count the number of bound states of the spatial operator.

5 Conclusions

In this article we have studied the linear stability of soliton and hairy black hole solutions of $\mathfrak{su}(2)$ Einstein-Yang-Mills theory with a negative cosmological constant. We have applied a recently developed perturbation formalism which is based on curvature quantities. The advantage of this approach is that one can work with gauge-invariant quantities, which are governed by a wave equation whose spatial part is symmetric. As a consequence, we were able to decouple the constraint variables by a supersymmetric-like transformation, and to show analytically the linear stability of soliton and black holes with respect to odd-parity fluctuations. We stress that such a proof is unlikely to exist in a gauge-invariant metric formulation, since when non-Abelian gauge fields are coupled to the metric, the metric approach fails to yield a symmetric wave operator in a natural way.

Our main result concerns those solutions in which the function w which determines the gauge field potential has no zeros and the cosmological constant is large and negative. These solutions are of particular interest as it was already known that they have no modes of instability under spherically symmetric perturbations. We have proved that this holds also for general, linear, perturbations in the odd-parity sector. This result is significant because it shows that these static configurations possess no topological instabilities. Therefore the presence of a sufficiently large (negative) cosmological constant (which means that there is a large gravitational potential away from the black hole horizon, or the centre of the solitons) stabilizes the situation. This may be understood heuristically by analogy with the situation in quantum field theory in curved space. A black hole can be in thermal equilibrium with a bath of radiation at the Hawking temperature, however in asymptotically flat space this equilibrium is unstable. Stability can be restored either by placing the whole system in a box, or if the black hole instead resides in asymptotically anti-de Sitter space [23], provided the cosmological constant is sufficiently large and negative. In some sense, then, the structure of asymptotically adS space plays a similar role in our situation to placing the black hole or soliton in a box, which means that the gauge field cannot escape to infinity, as happens when the corresponding configurations in asymptotically flat space are perturbed [24]. Further, because of the above stability of the Hartle-Hawking state, the black holes we have studied in this paper are precisely those which are most of interest from the quantum field theory point of view. Therefore it is important to establish their classical stability before studying the properties of quantum fields propagating on these geometries. This will be the subject of future work.

Classically, some open questions remain. Firstly, we need to investigate the stability of these solutions under non-spherically symmetric even-parity perturbations. The even-parity is generally less amenable to analysis, since its properties depend crucially on the detailed structure of the equilibrium configurations. However, in the situation in which we are interested, when the

cosmological constant is large and negative, it was found in [8] that, for spherically symmetric perturbations, the even-parity sector simplified considerably and stability was proved analytically in this case. Therefore it seems reasonable to suppose that simplification may be possible also for non-spherically symmetric perturbations.

Next, we have not examined the zero modes of the pulsation equations, i.e. the stationary solutions to the perturbation equations. Similarly to the asymptotically flat case [25], we except to find slowly rotating and charged solitons and black holes, the latter ones generalizing the Kerr-adS metric.

We shall return to these questions in a subsequent publication.

Acknowledgments

O.S. would like to thank H. Morales-Técotl and M. Tiglio for helpful discussions. This work was supported in part by the Swiss National Science Foundation.

A Harmonic decomposition of the wave operator

With respect to the background 3-metric

$$\bar{\boldsymbol{g}} = \frac{dr^2}{N} + r^2 d\Omega^2,$$

odd-parity perturbations of the extrinsic curvature can be expanded according to

$$L_{rr} = 0, \quad L_{rB} = \frac{\tilde{h}}{\sqrt{N}}\, S_B, \quad L_{AB} = r\tilde{k}\, 2\hat{\nabla}_{(A} S_{B)}\,. \tag{19}$$

Here and in the following, capital indices refer to coordinates on the 2-sphere. In the above equation, $S_B = S_B^{\ell m}$ are the transverse vector harmonics, and can be expressed in terms of the standard spherical harmonics $Y \equiv Y^{\ell m}$ as $S_B = (\hat{*}dY)_B$, where a hat refers to the standard metric on the 2-sphere.

Similarly, the electric YM field is expanded into $\mathfrak{su}(2)$-valued vector harmonics with odd parity (see [20] for details),

$$\begin{aligned}
\mathcal{E}_r &= \frac{\tilde{a}}{r\sqrt{N}}\, X_1 + \frac{\tilde{b}}{r\sqrt{N}}\, X_2, \\
\mathcal{E}_A &= \tilde{c}\,\tau_r Y_A + \tilde{d}\, Y\tau_A + \tilde{e}\left(\hat{\nabla}_A X_2 + \frac{1}{2}\mu^2 Y \tau_A\right),
\end{aligned} \tag{20}$$

where in terms of the Pauli matrices $\underline{\sigma} = (\sigma^i)$, $\tau_r = \underline{e}_r \cdot \underline{\sigma}/(2i)$, $\tau_A = \underline{e}_A \cdot \underline{\sigma}/(2i)$. Here X_1 and X_2 are the $\mathfrak{su}(2)$-valued harmonics

$$X_1 = Y\,\tau_r, \quad X_2 = \hat{g}^{AB}\hat{\nabla}_A Y \tau_B\,.$$

The tensor harmonics in the expansions (19) and (20) are chosen to be orthogonal. After the rescaling

$$\tilde{h} = n_1 h, \quad \tilde{k} = n_2 k, \quad \tilde{a} = m_1 a, \quad \tilde{b} = m_1 b, \quad \tilde{c} = m_1 c, \quad \tilde{d} = m_1 d, \quad \tilde{e} = m_1 e,$$

where in terms of $\mu^2 = \ell(\ell+1)$ and $\lambda = \mu^2 - 2$, the coefficients are given by

$$n_1 = \frac{1}{\sqrt{2\mu^2}}, \quad n_2 = \frac{1}{\sqrt{2\mu^2\lambda}},$$
$$m_1 = \frac{1}{\sqrt{2G}}, \quad m_2 = m_3 = \frac{1}{\sqrt{2G\mu^2}}, \quad m_4 = \frac{1}{\sqrt{4G}}, \quad m_5 = \frac{1}{\sqrt{G\mu^2\lambda}},$$

the expansion is normalized such that

$$\langle (L,\mathcal{E})\,,\,(L,\mathcal{E})\rangle = \int (h^2 + k^2 + a^2 + b^2 + c^2 + d^2 + e^2)\frac{dr}{\sqrt{N}}\,,$$

where $\langle .\,,\,.\rangle$ denotes the scalar product defined by (5). Note that for $\ell = 1$, $\hat{\nabla}_{(A}S_{B)}$ and $\hat{\nabla}_A X_2 + \frac{1}{2}\mu^2 Y\tau_A$ vanish, and therefore, the amplitudes k and e do not exist in those cases. For $\ell = 0$, the function Y is constant and only the amplitudes a and d are present.

An efficient way to perform the harmonic decomposition of the wave operator is to compute the energy expression which corresponds to $\hat{\Lambda}_{ij}$ and $\Lambda^{(YM)}$,

$$E = E_{grav} + E_{YM} + E_{int},$$

where

$$\begin{aligned}
E_{grav} \;&=\; \frac{1}{2}\int\Bigg(\frac{1}{\alpha^2}\dot{L}^{ij}\cdot\dot{L}_{ij} + \bar{\nabla}^k L^{ij}\cdot\bar{\nabla}_k L_{ij} + 2L^{ij}\bar{R}^k_i L_{jk} - 2L^{ij}\bar{R}_{kilj}L^{kl} \\
&\quad + \frac{8}{\alpha}L^{ij}\bar{\nabla}_i\left(\alpha^k L_{jk}\right) - 2L^{ij}\bar{\nabla}^k\left(\frac{\alpha_i}{\alpha}\right)L_{jk} - \frac{L^{ij}}{\alpha^2}\bar{\nabla}_i\alpha^2\bar{\nabla}_j\left(\frac{A}{\alpha}\right) \\
&\quad -2\Lambda L^{ij}L_{ij} + 4G\,\mathrm{Tr}\left\{L^{ij}\bar{F}^k_{\ i}\bar{F}^l_{\ j}L_{kl} + \frac{1}{4}\bar{F}_{kl}\bar{F}^{kl}L_{ij}L^{ij}\right\}\Bigg)\alpha\sqrt{\bar{g}}\,d\,x^3, \\
E_{YM} \;&=\; G\int\mathrm{Tr}\Bigg\{\frac{1}{\alpha^2}\dot{\mathcal{E}}^i\dot{\mathcal{E}}_i + 2\bar{D}^{[i}\mathcal{E}^{j]}\cdot\bar{D}_{[i}\mathcal{E}_{j]} + \alpha^2\left[\bar{D}^j\left(\frac{\mathcal{E}_j}{\alpha}\right)\right]^2 \\
&\quad + \bar{F}^{ij}[\mathcal{E}_i,\mathcal{E}_j] + 4G\,\mathrm{Tr}(\bar{F}^l_{\ k}\mathcal{E}_l)\bar{F}^{ik}\mathcal{E}_i\Bigg\}\alpha\sqrt{\bar{g}}\,d\,x^3, \\
E_{int} \;&=\; -4G\int\mathrm{Tr}\Bigg\{L^{ij}\bar{F}_i^{\ k}\bar{D}_k\mathcal{E}_j + \frac{1}{\alpha^2}L^{ij}\mathcal{E}_k\bar{D}_i\left(\alpha^2\bar{F}_j^{\ k}\right) \\
&\quad + \frac{1}{2}\mathcal{E}^i\bar{F}_{ij}\bar{\nabla}^j\left(\frac{A}{\alpha}\right)\Bigg\}\alpha\sqrt{\bar{g}}\,d\,x^3.
\end{aligned}$$

For a spherically symmetric background with no electric field, non-vanishing background quantities are given by

$$\bar{R}^r_{\ ArB} = -\tfrac{r}{2}N_r\hat{g}_{AB}, \quad \bar{R}^D_{\ CAB} = 2(1-N)\delta^D_{[A}\hat{g}_{B]C},$$

$$\bar{R}_{rr} = -\tfrac{N_r}{rN}, \qquad \bar{R}_{AB} = \left(1 - N - \frac{r}{2}N_r\right)\hat{g}_{AB},$$
$$\bar{F}_{rB} = -w_r\,\hat{\varepsilon}^A_{\ B}\tau_A, \quad \bar{F}_{AB} = (w^2-1)\tau_r\,\hat{\varepsilon}_{AB}.$$

Using this, the background equations and the expansions (19) and (20), the energy expression yields, after integrating over the spherical variables,

$$E = \frac{1}{2}\int\Big\{(\dot{U},\dot{U}) + (\partial_\rho U, \partial_\rho U) + 2(U, \boldsymbol{A}\partial_\rho U) + (U, \boldsymbol{S}U)\Big\}d\rho,$$

where $U = (h, a, b, k, c, d, e)$, and where $\boldsymbol{A}$ and $\boldsymbol{S}$ are given in section 4. The radial coordinate ρ is defined by $\partial_\rho = SN\partial_r$, and $(.\,,.)$ denotes the standard scalar product. The wave equation (8) follows directly from this energy functional.

B The initial value formulation

In this appendix, we give a gauge-invariant formulation of the initial value problem. In order to solve the linearized EYM equations, we have to take into account the Hamiltonian constraint and all evolution equations which we had differentiated in time. These equations cannot be described in terms of the linearized extrinsic curvature and the electric YM field alone, since perturbations of the 3-metric and the magnetic gauge potential appear (and not only their time derivatives). However, if the background is spherically symmetric, one can give a formulation in terms of gauge-invariant quantities which we introduced in an earlier article [20].

It turns out that the "missing" constraint equations (i.e. the linearized Hamiltonian constraint and the relevant evolution equations we had differentiated in time) are equivalent to the ρ-components of the equations (41), (42) and (43), and to equation (46) of Ref. [20] [The presence of the cosmological constant does not modify those perturbation equations.] Using the gauge-invariant amplitudes H, A, B and C defined there, these equations become

$$\begin{aligned}
2\partial_t\tilde{h} &= -\lambda\frac{\gamma}{r}H_\rho + \frac{4G\gamma}{r}\left[(w^2-1)(A_\rho - wB_\rho) - w_\rho C - \frac{(1-w^2)^2}{r^2}H_\rho\right],\\
\partial_t\tilde{a} &= \gamma\left[(\mu^2+2w^2)A_\rho - 2\mu^2 wB_\rho - 2wC_\rho + 2w_\rho C + \mu^2(1-w^2)\frac{H_\rho}{r^2}\right],\\
\partial_t\tilde{b} &= \gamma\left[-2wA_\rho + (\mu^2+w^2-1)B_\rho + C_\rho - w(1-w^2)\frac{H_\rho}{r^2}\right],\\
\partial_t\left(\tilde{d} + \frac{1}{2}\mu^2\,\tilde{e}\right) & \\
&= w\partial_\rho A_\rho + 2w_\rho A_\rho + \gamma^2[\mu^2+w^2-1]C - C_{\rho\rho} + \mu^2 w_\rho\frac{H_\rho}{r^2}\,. \qquad (21)
\end{aligned}$$

Furthermore, one has the relation between the gauge-invariant amplitudes of Ref. [20] and the curvature-based amplitudes $\tilde{h}$, $\tilde{k}$, $\tilde{a}$, ... introduced in appendix

A of this article. For $\ell \geq 2$, one has

$$
\begin{aligned}
H_t &= -2r\tilde{k}, & \dot{H}_\rho &= 2\alpha\tilde{h} - 2r^2\partial_\rho\left(\frac{\tilde{k}}{r}\right), \\
A_t &= \left(\tilde{c} + w\tilde{e} - 2\tfrac{w^2-1}{r}\tilde{k}\right), & \dot{A}_\rho &= -\gamma\tilde{a} + \partial_\rho A_t\,, \\
B_t &= \tilde{e}, & \dot{B}_\rho &= -\gamma\tilde{b} + \partial_\rho\tilde{e} - 2\frac{w_\rho}{r}\tilde{k}, \\
\dot{C} &= -\tilde{d} - \tfrac{1}{2}\mu^2\tilde{e} + wA_t\,. & &
\end{aligned}
\tag{22}
$$

The initial value problem for $\ell \geq 2$ can be solved as follows. First, we choose any functions $H_\rho = H_\rho^{(0)}$, $A_\rho = A_\rho^{(0)}$, $B_\rho = B_\rho^{(0)}$ and $C = C^{(0)}$ on an initial time slice, $\Sigma_{t=0}$, say. Next, one solves the momentum constraint equations (9) for $U = (h, a, b, k, c, d, e)$ on the initial time slice. A convenient way to do this is to freely specify the functions c, d, e and k and to compute h, a and b using (9). Then, the time derivative of U on Σ_0 is consistently given by equations (21) and the time derivatives of the momentum constraint equations. The amplitudes U are then evolved using the symmetric wave equation (8). Finally, the gauge-invariant amplitudes H, A, B and C, parametrizing the metric and the gauge potential are obtained from (22) after integration over t, where the integration "constants" are given by $H_\rho^{(0)}$, $A_\rho^{(0)}$, $B_\rho^{(0)}$ and $C^{(0)}$. For $\ell = 1$, the perturbation equations can be solved in a similar manner, but one has to take into account that $C = 0$ and that the relation (22) is different in that case.

Since we have shown that the pulsation equations admit no solutions which grow exponentially in time when $|\Lambda|$ is large enough, the relations (22) show that the same must hold for the gauge-invariant quantities H, A, B, and C, parametrizing the metric and the gauge-potential. This completes our stability analysis.

References

[1] R. Bartnik and J. MacKinnon, Phys. Rev. Lett. **61**, 141 (1988).

[2] M.S. Volkov and D.V. Gal'tsov, JETP Lett. **50**, 346 (1989);
H.P. Künzle and A.K.M. Masood-ul-Alam, J. Math. Phys. **31**, 928 (1990);
P. Bizon, Phys. Rev. Lett. **64**, 2844 (1990).

[3] M.S. Volkov and D.V. Gal'tsov, Phys. Reps. **319**, 2 (1999).

[4] O. Brodbeck and N. Straumann, J. Math. Phys. **37**, 1414 (1996).

[5] D.V. Gal'tsov and M.S. Volkov, Phys. Lett. **B273**, 255 (1991).

[6] O. Brodbeck, M. Heusler, G. Lavrelashvili, N. Straumann and M.S. Volkov, Phys. Rev. **D54**, 7338 (1996).

[7] J. Bjoraker and Y. Hosotani, Phys. Rev. Lett. **84**, 1853 (2000);
J. Bjoraker and Y. Hosotani, Phys. Rev. **D62**, 043513 (2000).

[8] E. Winstanley, Class. Quantum Grav. **16**, 1963 (1999).

[9] O. Sarbach, *On the generalization of the Regge-Wheeler equation for self-gravitating matter fields*, PhD thesis (unpublished).

[10] J. Baacke and H. Lange, Mod. Phys. Lett. **A7**, 1455 (1992).

[11] O. Brodbeck, M. Heusler, and O. Sarbach, Phys. Rev. Lett. **84**, 3033 (2000).

[12] A.A. Ershov and D.V. Gal'tsov, Phys. Lett. **A150**, 159 (1990).

[13] P. Breitenlohner, P. Forgacs and D. Maison, Comm. Math. Phys. **163**, 141 (1994).

[14] M.S. Volkov, N. Straumann, G. Lavrelashvili, M. Heusler and O. Brodbeck, Phys. Rev. **D54**, 7243 (1996).

[15] P. Kanti and E. Winstanley, Phys. Rev. **D61**, 084032 (2000).

[16] O. Sarbach, M. Heusler and O. Brodbeck, *Self-adjoint wave equations for dynamical perturbations of self-gravitating fields*, gr-qc/0010067; to appear in Phys. Rev. **D**.

[17] A. Anderson, A.M. Abrahams, and C. Lea, Phys. Rev. **D58**, 064015 (1998).

[18] T. Regge and J. Wheeler, Phys. Rev. **108**, 1063 (1957).

[19] M.S. Volkov, in *Proceedings of the Workshop "Geometry and Integrable Models"*, edited by P.N. Pyatov and S.N. Solodukhin, pages 55-77 (World Scientific, 1996).

[20] O. Sarbach, M. Heusler, and O. Brodbeck, Phys. Rev. **D62**, 084001 (2000).

[21] M.S. Volkov and D.V. Gal'tsov, Phys. Lett. B **341**, 279 (1995).

[22] H. Amann and P. Quittner, J. Math. Phys. **36**, 4553 (1995).

[23] S.W. Hawking and D.N. Page, Comm. Math. Phys. **87**, 577 (1983).

[24] Z.-H. Zhou and N. Straumann, Nucl. Phys. **B360**, 180 (1991).

[25] M.S. Volkov and N. Straumann, Phys. Rev. Lett. **79**, 1428 (1997);
O. Brodbeck, M. Heusler, N. Straumann, and M.S. Volkov, Phys. Rev. Lett. **79**, 4310 (1997).